An Introduction to Electromagnetic Source Estimation

Forward Solution

Start with Maxwell's equations and charge conservation.

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathbf{t}}$$

$$\nabla \times \mathbf{B} = \mu_{\mathbf{0}}(\mathbf{J} + \epsilon_{\mathbf{0}} \frac{\partial \mathbf{E}}{\partial \mathbf{t}})$$

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial \mathbf{t}} = \mathbf{0}$$

Define potentials from which the fields can be computed.

$$\mathbf{E} = -(\nabla \phi + \frac{\partial \mathbf{A}}{\partial \mathbf{t}})$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Cases of biological interest are *quasineutral* ($\rho \simeq 0$) and *quasistatic* ($\frac{\partial}{\partial t} = 0$). Maxwell's equations for this case are as follows.

$$\nabla \cdot \mathbf{E} = -\nabla^{\mathbf{2}} \phi \simeq \mathbf{0}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{E} = \mathbf{0}$$

$$\nabla \times \mathbf{B} = \mu_{\mathbf{0}} \mathbf{J}$$

$$\nabla \cdot \mathbf{J} = \mathbf{0}$$

The choice of quasineutrality means that we can't use Poisson's equation to find the electric field. Assume that the current comes in two parts – the first part, caused by neural action is called the *impressed current* J_i and the second part is the *return current* described by Ohm's law.

$$\mathbf{J} = \mathbf{J}_i + \sigma \mathbf{E} \tag{1}$$

$$\nabla \cdot \mathbf{J} = \nabla \cdot \mathbf{J}_i - \nabla \cdot (\sigma \nabla \phi) = 0$$

$$\nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \mathbf{J}_i \tag{2}$$

This is our version of Poisson's equation, except that instead of the charge density serving as a source for $\mathbf{E}$ we have the divergence of the impressed current.

Solution for Simple Geometry

Consider the case of a uniform sphere of radius R and conductivity σ with a single dipole source. We need to find the potential on the surface of the sphere, since that's what the EEG will measure. If the medium is *linear* we can compute the potential from a single source and the add up the contribution from all the sources to get the actual potential. Let the source be a *current dipole* – a source of magnitude $+\dot{q}$ separated by $\mathbf{d}$ from a sink with $-\dot{q}$. The current dipole is $\mathbf{Q} = \dot{q}\mathbf{d}$. For our case consider a point dipole, where one takes the limit as $d \to 0$ and $\dot{q} \to \infty$ in such a way that Q is constant. Place the dipole on the z axis – it's easy to generate the values at any other location using a simple rotation. Let's first consider the case where the dipole points in the $\hat{\mathbf{z}}$ direction : $\mathbf{J}_i = Q_z \hat{\mathbf{z}} \delta(z-b)\delta(x)\delta(y)$.

$$\sigma \nabla^2 \phi = Q_z \nabla \cdot (\hat{\mathbf{z}} \delta(z-b)\delta(x)\delta(y))$$

The solution (ϕ) is the sum of the particular solution ϕ_p and the general solution to $\nabla^2 \phi_g = 0$. For the case of a dipole with moment $\mathbf{Q}$ located at a location $\mathbf{b}$

$$\phi_p = \frac{1}{4\pi\sigma} \frac{\mathbf{Q} \cdot (\mathbf{r} - \mathbf{b})}{|\mathbf{r} - \mathbf{b}|^3}$$

Use the multipole expansion in Legendre polynomials $P_l(x)$ (See for instance [PLL88, §5.4, p89ff]).

$$\phi_p(r,\theta) = \frac{Q_z}{4\pi\sigma} \frac{1}{rb} \sum_{l=0}^{\infty} l \left(\frac{b}{r}\right)^l P_l(\cos\theta)$$

If one considers the symmetry of the situation ($\partial/\partial\phi = 0$), the general solution to Laplace's equation is

$$\phi_g(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l \frac{1}{r^{l+1}}\right] P_l(\cos\theta)$$

To reach the final solution we must impose boundary conditions on the problem.

- ϕ is continuous at $r = R$.
- ϕ is bounded as $r \to 0$.
- $\left[\frac{\partial \phi}{\partial r}\right]_R = 0$

The second boundary condition implies that $B_l = 0$. Form the full solution $\phi = \phi_p + \phi_g$, take the normal derivative at the surface of the sphere and set that equal to zero.

$$0 = \sum_l \left[l A_l R^{l-1} + \frac{Q_z}{4\pi\sigma} \frac{l}{b} - \frac{(l+1)b^l}{R^{l+2}}\right] P_l(\cos\theta)$$

Because the Legendre polynomials are orthogonal the quantity in the brackets must be zero for each l.

$$A_l = \frac{Q_z}{4\pi\sigma} \frac{(l+1)b^{l-1}}{R^{2l+1}}$$

Insert this back into the general equation to get a final result for the potential from a radial source on the z axis.

$$\phi(r,\theta) = \frac{Q_z}{4\pi\sigma R^2} \sum_{l=0}^{\infty} \left(\frac{b}{R}\right)^{l-1} (2l+1) P_l(\cos\theta) \quad (3)$$

We can carry out the same procedure for a transverse dipole located on the z axis, where $\mathbf{J} = (Q_x\hat{\mathbf{x}} + Q_y\hat{\mathbf{y}})\delta(z-b)\delta(x)\delta(y)$. The solution for this source is

$$\begin{aligned} \phi(r,\theta,\varphi) = & \left(\frac{Q_x\cos\varphi + Q_y\sin\varphi}{4\pi\sigma R^2}\right) \times \\ & \left(\sum_{l=1}^{\infty} \frac{2l+1}{l} \left(\frac{b}{R}\right)^{l-1} P_l^1(\cos\theta)\right) \end{aligned} \quad (4)$$

($P_l^1(x)$ is the first associated Legendre polynomial.) So, we can now write down the potential at a point on the surface of the sphere $\Omega = (\theta, \varphi)$ associated with a single equivalent current dipole source of magnitude $\mathbf{Q}$ at location $b\hat{\mathbf{z}}$.

$$\phi(\Omega) = \mathbf{A}(\Omega, \mathbf{r}) \cdot \mathbf{Q}$$

$\mathbf{A}(\Omega, \mathbf{r})$ is the non-linear function of position derived above. (The function will be different for different geometries but the overall principles remain the same. See [MP95] or [Nun81] for a detailed listing of the results from different source distributions and conductivities.) Having done this, it's easy to write down the potential at a point as a result of several sources.

$$\phi(\Omega) = \mathbf{A}(\Omega, \mathbf{r}_1) \cdot \mathbf{Q}_1 + \mathbf{A}(\Omega, \mathbf{r}_2) \cdot \mathbf{Q}_2 + \ldots$$

Similarly, one can write down a system of equations that describes the voltages at a set of locations, typically the electrodes as a function of a number of sources.

$$\phi_1 = \phi(\Omega_1) = \mathbf{A}(\Omega_1, \mathbf{r}_1) \cdot \mathbf{Q}_1 + \mathbf{A}(\Omega_1, \mathbf{r}_2) \cdot \mathbf{Q}_2 + \ldots$$

$$\phi_2 = \phi(\Omega_2) = \mathbf{A}(\Omega_2, \mathbf{r}_1) \cdot \mathbf{Q}_1 + \mathbf{A}(\Omega_2, \mathbf{r}_2) \cdot \mathbf{Q}_2 + \ldots$$

In matrix notation we write the matrix which describes the coupling between the j'th source and the i'th measurement location as $\mathsf{A} = \{a_{ij}\} = \mathsf{A}(\Omega_i, r_j)$. Call the measured voltages $\mathsf{V} = \{\phi_i\}$ and the sources $\mathsf{q} = \{Q_j\}$.

$$\mathsf{V} = \mathsf{A} \cdot \mathsf{q} \quad (5)$$

Source Estimation

We have now solved the problem to some extent. We have written down a set of linear equations relating the sources to the observed voltages, so all that remains is to solve the equations, inverting the matrix operator A and finding the sources in terms of the observed voltages ($\mathsf{q} = \mathsf{A}^{-1} \cdot \mathsf{V}$). Unfortunately, this is a system with ~ 100 equations (the number of electrodes) and something like ~ 100 billion unknowns (the number of the neurons in the brain). Even if we consider that groups of neurons tend to fire together the number of possible sources is still much greater than the number of equations. Obviously, some rethinking of the issues is required.

Inverse Problems

Most of the problems we face in physics are of the sort solved in the previous section. We start with general laws, often in the form of differential or integral equations and a set of initial and/or boundary conditions and proceed to find the solution. Such problems are almost always *well posed*. This means that they obey the following criteria.

- Solutions are unique.
- Solutions exist for any data.
- Solutions are *stable* - small changes in the initial conditions lead to small changes in the solution.

Unfortunately, inverse problems such as ours fail these tests - *Inverse problems are generally ill-posed.* The forward solution, the typical case, involves starting with the maximal amount of information and working forward. For instance, we start with Newton's laws and the initial positions and velocities of the planets and solve for their subsequent trajectories. When Newton first solved his inverse problem what he did was much harder – he started with the tabulated positions of the planets and had to derive the laws of motion. Inverse problems are quite common, appearing in fields such as geophysics [Par94] and image processing [BB98]. There are various reason why the source estimation problem is ill-posed. Some sets of sources fall in the null space of A – e.g. $\mathsf{A} \cdot \mathsf{q} = 0$ so they will never show up in the output voltages no matter what their magnitude. Alternatively, when noise is taken into account the measured values $\mathsf{V} + \nu$ may not fall in the range of A at all so that no possible sources exist which will provide the measured results.

Clearly, we have to shift our thinking from finding *the* solution to finding the *best* solution in some possible sense. The ways in which we define "best" determines the approach we take.

Non-linear Equivalent Current Dipole Fit

One approach is to reduce the possible number of sources. The approach used in EEG when analyzing Event Related Potentials (ERPs) is to measure a large number of events, synchronize them using the triggering event and average them together. This causes the source of interest to stand out and other brain activity to average to zero. So, it's reasonable to try to fit the data with a small number of equivalent current dipoles – recall that each dipole requires six parameters, three for position and three for the direction and magnitude of $\mathbf{Q}$. So, for a reduced source space Q^* we want to find the value of Q^* which minimizes the value of

$$\min_{\mathsf{Q}^*} |\mathsf{A} \cdot \mathsf{Q}^* - \mathsf{V}| \tag{6}$$

How many dipoles should we use? Obviously, it makes no sense to try to fit more parameters than the data contains. (You can run a quadratic polynomial through two points, but it isn't very meaningful.) So, how can we estimate how much information we have? Use *Singular Value Decomposition* ([Lip68], [WHPV92, §2.6, 15.4]) to decompose the matrices of interest as follows.

$$\mathsf{A} = \mathsf{UWV}^T$$

$$\mathsf{W} = \mathrm{diag}(w_1, w_2, \ldots, w_N)$$

The values w_i are the *singular values* of A. The number of singular values which are significantly larger than zero gives the *rank* of A, which tells us the number of pieces of information A really contains.
There are two relevant matrices to consider. The first involves the relationship between the sources and electrodes we computed above($\mathsf{A} = \mathsf{A}(\Omega_i, r_j)$). This tells us how many of the sources we can actually expect to see with the electrode positions used for the experiment. The second matrix is the covariance of the measured voltages. There is usually a significant amount of correlation between the electrodes, since the sources can potentially have an effect at each electrode. The rank of these matrices tells us how much information we have on hand and so tells us how many dipoles we should be able to fit. The number is usually fairly small (< 10).

Use your favorite non-linear optimising method to find the best values for Q^* and you have a good idea where the sources are that account for the observed voltages on the scalp.

Dipole Lattices

The other approach is to try to solve the equation "as is." For instance, create a three dimensional grid of 1 centimeter cubes and place a dipole on each vertex. This gives us roughly 6000 values to find, since the brain has a volume of $\sim 2000\text{cm}^3$ and three dipole magnitudes are required at each location. Again, singular value decomposition can do this for us, since it allows us to explicitly construct the inverse matrix of A.

$$\mathsf{A}^{-1} = V \cdot \mathrm{diag}(1/w_1, 1/w_2, \ldots, 1/w_N) \cdot \mathsf{U}^T$$

(Note that if any of the $w_j = 0$ the matrix is singular.) Regularize this by setting $1/w_j$ to 0 when $|w_j| < \epsilon$. In effect this throws away solutions which are dominated by noise (both physical and numerical) and keeps that part of the matrix which belongs to our data. The matrix is still singular, but we're no longer trying to balance vectors which are (almost) in the null space of A. Call this normalized inverse matrix A^+. This produces an estimate for the source

$$\mathsf{Q}^* = \mathsf{A}^+ \cdot \mathsf{V} \tag{7}$$

The solution is no longer unique, since it's possible to add any vector from the null space of A to Q^* without changing the predicted voltage. We need to make it unique by specifying *a priori* information - conditions which we wish to impose apart from any particular set of data. Possible candidates for this are

- $|\mathsf{Q}^*|$ is a minimum.
- Q^* is the *smoothest* possible solution.
- Q^* has the best correlation with some other form of functional imaging (PET or fMRI).

These constraints allow us to select a particular solution from the family of solutions specified above. An additional form of constraint can be taken from brain anatomy. The majority of the sources are located on the cortical surface oriented normal to that surface. We can use image processing techniques on MRIs of the head to find the cortical surface and approximate it with triangles. The vertices of the triangles provide the potential source locations. This has the advantage that we're only placing sources in physically plausible locations and orientations. The disadvantage is that it takes quite a few triangles (tens of thousands) to produce a reasonable approximation to the cortex.

References

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